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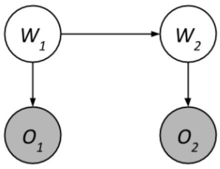
Artificial Intelligence II

Exercise 6

Q1. HMMs

Consider the following Hidden Markov Model.



W_1	$P(W_1)$
0	0.3
1	0.7

W_t	W_{t+1}	$P(W_{t+1}\|W_t)$
0	0	0.4
0	1	0.6
1	0	0.8
1	1	0.2

W_t	O_t	$P(O_t\|W_t)$
0	A	0.9
0	B	0.1
1	A	0.5
1	B	0.5

Suppose that we observe $O_1 = A$ and $O_2 = B$.
Using the forward algorithm, compute the probability distribution $P(W_2|O_1 = A, O_2 = B)$ one step at a time.

1. Compute $P(W_1, O_1 = A)$.

Solution:

$P(W_1, O_1 = A) = P(W_1)P(O_1 = A|W_1)$
$P(W_1 = 0, O_1 = A) = (0.3)(0.9) = 0.27$
$P(W_1 = 1, O_1 = A) = (0.7)(0.5) = 0.35$

2. Using the previous calculation, compute $P(W_2, O_1 = A)$.

Solution:

$P(W_2, O_1 = A) = \sum_{x_1} P(x_1, O_1 = A)P(W_2|x_1)$
$P(W_2 = 0, O_1 = A) = (0.27)(0.4) + (0.35)(0.8) = 0.388$
$P(W_2 = 1, O_1 = A) = (0.27)(0.6) + (0.35)(0.2) = 0.232$

3. Using the previous calculation, compute $P(W_2, O_1 = A, O_2 = B)$.

Solution:

$P(W_2, O_1 = A, O_2 = B) = P(W_2, O_1 = A)P(O_2 = B|W_2)$
$P(W_2 = 0, O_1 = A, O_2 = B) = (0.388)(0.1) = 0.0388$
$P(W_2 = 1, O_1 = A, O_2 = B) = (0.232)(0.5) = 0.116$

4. Finally, compute $P(W_2|O_1 = A, O_2 = B)$.

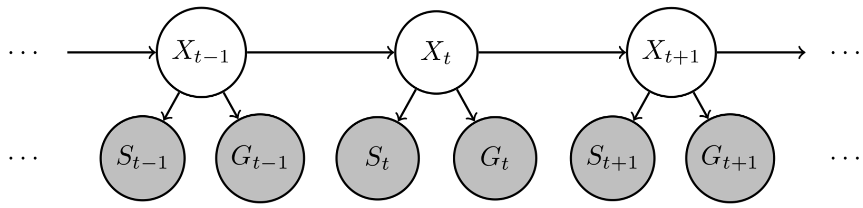
Solution:

Renormalizing the distribution above, we have
$P(W_2 = 0|O_1 = A, O_2 = B) = 0.0388/(0.0388 + 0.116) \approx 0.25$
$P(W_2 = 1|O_1 = A, O_2 = B) = 0.116/(0.0388 + 0.116) \approx 0.75$

Q2. HMM: Where is the Car?

Transportation researchers are trying to improve traffic in the city but, in order to do that, they first need to estimate the location of each of the cars in the city. They need our help to model this problem as an inference problem of an HMM. For this question, assume that only *one* car is being modeled.

(a) The structure of this modified HMM is given below, which includes X, the location of the car; S, the noisy location of the car from the signal strength at a nearby cell phone tower; and G, the noisy location of the car from GPS.



We want to perform filtering with this HMM. That is, we want to compute the belief $P(x_t|s_{1:t}, g_{1:t})$, the probability of a state x_t given all past and current observations.

The **dynamics update** expression has the following form:

$$P(x_t|s_{1:t-1}, g_{1:t-1}) = \underline{\quad \textbf{(i)} \quad} \; \underline{\quad \textbf{(ii)} \quad} \; \underline{\quad \textbf{(iii)} \quad} \; P(x_{t-1}|s_{1:t-1}, g_{1:t-1})$$

Complete the expression by choosing the option that fills in each blank.

(i)
- ☐ $P(s_{1:t}, g_{1:t})$
- ☐ $P(s_{1:t-1}, g_{1:t-1})$
- ☐ $P(s_{1:t-1})P(g_{1:t-1})$
- ☐ $P(s_{1:t})P(g_{1:t})$
- ■ 1

(ii)
- ☐ $\sum_{x_t}$
- ■ $\sum_{x_{t-1}}$
- ☐ $max_{x_{t-1}}$
- ☐ max_{x_t}
- ☐ 1

(iii)
- ☐ $P(x_{t-1}|x_{t-2})$
- ☐ $P(x_{t-2}, x_{t-1})$
- ☐ $P(x_{t-1}, x_t)$
- ■ $P(x_t|x_{t-1})$
- ☐ 1

Solution:

The derivation of the dynamics update is similar to the one for the canonical HMM, but with two observation variables instead.

$$\begin{aligned} P(x_t|s_{1:t-1}, g_{1:t-1}) &= \sum_{x_{t-1}} P(x_{t-1}, x_t|s_{1:t-1}, g_{1:t-1}) \\ &= \sum_{x_{t-1}} P(x_t|x_{t-1}, s_{1:t-1}, g_{1:t-1})P(x_{t-1}|s_{1:t-1}, g_{1:t-1}) \\ &= \sum_{x_{t-1}} P(x_t|x_{t-1})P(x_{t-1}|s_{1:t-1}, g_{1:t-1}) \end{aligned}$$

In the last step, we use the independence assumption given in the HMM, $X_t \perp\!\!\!\perp S_{1:t-1}, G_{1:t-1}|X_{t-1}$.

The **observation update** expression has the following form:

$$P(x_t|s_{1:t}, g_{1:t}) = \underline{\quad \textbf{(iv)} \quad} \; \underline{\quad \textbf{(v)} \quad} \; \underline{\quad \textbf{(vi)} \quad} \; P(x_t|s_{1:t-1}, g_{1:t-1}).$$

Complete the expression by choosing the option that fills in each blank.

(iv)

- ☐ $P(s_{1:t-1}|s_t)P(g_{1:t-1}|g_t)$
- ☐ $P(s_t, g_t|s_{1:t-1}, g_{1:t-1})$
- ☐ $\frac{1}{P(s_t|s_{1:t-1})P(g_t|g_{1:t-1})}$
- ■ $\frac{1}{P(s_t,g_t|s_{1:t-1},g_{1:t-1})}$
- ☐ $P(s_{1:t-1}, g_{1:t-1}|s_t, g_t)$
- ☐ $\frac{1}{P(s_{1:t-1}|s_t)P(g_{1:t-1}|g_t)}$
- ☐ $\frac{1}{P(s_{1:t-1},g_{1:t-1}|s_t,g_t)}$
- ☐ $P(s_t|s_{1:t-1})P(g_t|g_{1:t-1})$
- ☐ 1

(v)

- ☐ $\sum_{x_t}$
- ☐ $\sum_{x_{t-1}}$
- ☐ max_{x_t}
- ☐ $max_{x_{t-1}}$
- ■ 1

(vi)

- ☐ $P(x_{t-1}, s_{t-1})P(x_{t-1}, g_{t-1})$
- ☐ $P(s_{t-1}|x_{t-1})P(g_{t-1}|x_{t-1})$
- ☐ $P(x_{t-1}|s_{t-1})P(x_{t-1}|g_{t-1})$
- ☐ $P(x_{t-1}, s_{t-1}, g_{t-1})$
- ☐ $P(x_t, s_t)P(x_t, g_t)$
- ■ $P(s_t|x_t)P(g_t|x_t)$
- ☐ $P(x_t|s_t)P(x_t|g_t)$
- ☐ $P(x_t, s_t, g_t)$
- ☐ 1

Solution:

Again, the derivation of the observation update is similar to the one for the canonical HMM, but with two observation variables instead.

$$\begin{aligned}
P(x_t|s_{1:t}, g_{1:t}) &= P(x_t|s_t, g_t, s_{1:t-1}, g_{1:t-1}) \\
&= \frac{1}{P(s_t, g_t|s_{1:t-1}, g_{1:t-1})} P(x_t, s_t, g_t|s_{1:t-1}, g_{1:t-1}) \\
&= \frac{1}{P(s_t, g_t|s_{1:t-1}, g_{1:t-1})} P(s_t, g_t|x_t, s_{1:t-1}, g_{1:t-1})P(x_t|s_{1:t-1}, g_{1:t-1}) \\
&= \frac{1}{P(s_t, g_t|s_{1:t-1}, g_{1:t-1})} P(s_t, g_t|x_t)P(x_t|s_{1:t-1}, g_{1:t-1}) \\
&= \frac{1}{P(s_t, g_t|s_{1:t-1}, g_{1:t-1})} P(s_t|x_t)P(g_t|x_t)P(x_t|s_{1:t-1}, g_{1:t-1})
\end{aligned}$$

In the second to last step, we use the independence assumption $S_t, G_t \perp\!\!\!\perp S_{1:t-1}, G_{1:t-1}|X_t$; and in the last step, we use the independence assumption $S_t \perp\!\!\!\perp G_t|X_t$.