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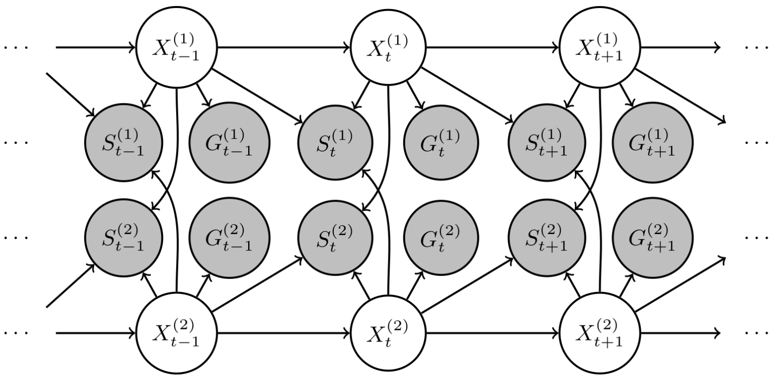
Artificial Intelligence II

Exercise 7

Q1. Particle Filtering: Where are the Two Cars?

As before, we are trying to estimate the location of cars in a city, but now, we model two cars jointly, i.e. car i for $i \in \{1, 2\}$. The modified HMM model is as follows:

- $X^{(i)}$ – the location of car i
- $S^{(i)}$ – the noisy location of the car i from the signal strength at a nearby cell phone tower
- $G^{(i)}$ – the noisy location of car i from GPS



d	$D(d)$	$E_L(d)$	$E_N(d)$	$E_G(d)$
-4	0.05	0	0.02	0
-3	0.10	0	0.04	0.03
-2	0.25	0.05	0.09	0.07
-1	0.10	0.10	0.20	0.15
0	0	0.70	0.30	0.50
1	0.10	0.10	0.20	0.15
2	0.25	0.05	0.09	0.07
3	0.10	0	0.04	0.03
4	0.05	0	0.02	0

The signal strength from one car gets noisier if the other car is at the same location. Thus, the observation $S_t^{(i)}$ also depends on the current state of the other car X_t, $j \neq i$.

The transition is modeled using a drift model D, the GPS observation $G_t^{(i)}$ using the error model E_G , and the observation $S_t^{(i)}$ using one of the error models E_L or E_N , depending on the car's speed and the relative location of both cars. These drift and error models are in the table above. **The transition and observation models are**:

$$P(X_t^{(i)}|X_{t-1}^{(i)}) = D(X_t^{(i)} - X_{t-1}^{(i)})$$

$$P(S_t^{(i)}|X_{t-1}^{(i)}, X_t^{(i)}, X_t^{(j)}) = \begin{cases} E_N(X_t^{(i)} - S_t^{(i)}) & \text{, if } |X_t^{(i)} - X_{t-1}^{(i)}| \geq 2 \text{ or } X_t^{(i)} = X_t^{(j)} \\ E_L(X_t^{(i)} - S_t^{(i)}) & \text{, otherwise} \end{cases}$$

$$P(G_t^{(i)}|X_t^{(i)}) = E_G(X_t^{(i)} - G_t^{(i)})$$

Throughout this problem you may give answers either as unevaluated numeric expressions (e.g. $0.1 \cdot 0.5$) or as numeric values (e.g. 0.05). The questions are decoupled.

(a) Assume that at $t = 3$, we have the single particle $(X_3^{(1)} = -1, X_3^{(2)} = 2)$.

(i) What is the probability that this particle becomes $(X_4^{(1)} = -3, X_4^{(2)} = 3)$ after passing it through the dynamics model?

Solution:

$$\begin{aligned} P(X_4^{(1)} = -3, X_4^{(2)} = 3 | X_3^{(1)} = -1, X_3^{(2)} = 2) &= P(X_4^{(1)} = -3 | X_3^{(1)} = -1) \cdot P(X_4^{(2)} = 3 | X_3^{(2)} = 2) \\ &= D(-3 - (-1)) \cdot D(3 - 2) \\ &= 0.25 \cdot 0.10 \\ &= 0.025 \end{aligned}$$

(ii) Assume that there are no sensor readings at t = 4. What is the joint probability that the *original* single particle (from $t = 3$) becomes $(X_4^{(1)} = -3, X_4^{(2)} = 3)$ and then becomes $(X_5^{(1)} = -4, X_5^{(2)} = 4)$?

Solution:

$$\begin{aligned} & P(X_4^{(1)} = -3, X_5^{(1)} = -4, X_4^{(2)} = 3, X_5^{(2)} = 4 | X_3^{(1)} = -1, X_3^{(2)} = 2) \\ = & P(X_4^{(1)} = -3, X_5^{(1)} = -4 | X_3^{(1)} = -1) \cdot P(X_4^{(2)} = 3, X_5^{(2)} = 4 | X_3^{(2)} = 2) \\ = & P(X_5^{(1)} = -4 | X_4^{(1)} = -3) \cdot P(X_4^{(1)} = -3 | X_3^{(1)} = -1) \cdot P(X_5^{(2)} = 4 | X_4^{(2)} = 3) \cdot \\ & P(X_4^{(2)} = 3 | X_3^{(2)} = 2) \\ = & D(-4 - (-3)) \cdot D(-3 - (-1)) \cdot D(4 - 3) \cdot D(3 - 2) \\ = & D(-1) \cdot D(-2) \cdot D(1) \cdot D(1) \\ = & 0.1 \cdot 0.25 \cdot 0.1 \cdot 0.1 = 0.00025 \end{aligned}$$

For the remaining of this problem, we will be using 2 particles at each time step.

(b) At $t = 6$, we have particles $[(X_6^{(1)} = 3, X_6^{(2)} = 0), (X_6^{(1)} = 3, X_6^{(2)} = 5)]$. Suppose that after weighting, resampling, and transitioning from $t = 6$ to $t = 7$, the particles become $[(X_7^{(1)} = 2, X_7^{(2)} = 2), (X_7^{(1)} = 4, X_7^{(2)} = 1)]$.

(i) At $t = 7$, you get the observations $S_7^{(1)} = 2, G_7^{(1)} = 2, S_7^{(2)} = 2, G_7^{(2)} = 2$. What is the weight of each particle?

Particle	Weight
$(X_7^{(1)} = 2, X_7^{(2)} = 2)$	0.0225
$(X_7^{(1)} = 4, X_7^{(2)} = 1)$	0.000105

Solution:

$$P(S_7^{(1)} = 2|X_6^{(1)} = 3, X_7^{(1)} = 2, X_7^{(2)} = 2) \cdot P(G_7^{(1)} = 2|X_7^{(1)} = 2) \cdot$$
$$P(X_7^{(2)} = 2|X_6^{(2)} = 0, X_7^{(2)} = 2, X_7^{(1)} = 2) \cdot P(G_7^{(2)} = 2|X_7^{(2)} = 2)$$
$$= E_N(X_7^{(1)} - S_7^{(1)}) \cdot E_G(X_7^{(1)} - G_7^{(1)}) \cdot E_N(X_7^{(2)} - S_7^{(2)}) \cdot E_G(X_7^{(2)} - G_7^{(2)})$$
$$= E_N(2-2) \cdot E_G(2-2) \cdot E_N(2-2) \cdot E_G(2-2)$$
$$= E_N(0) \cdot E_G(0) \cdot E_N(0) \cdot E_G(0)$$
$$= 0.3 \cdot 0.5 \cdot 0.3 \cdot 0.5 = 0.0225$$

$$P(S_7^{(1)} = 2|X_6^{(1)} = 3, X_7^{(1)} = 4, X_7^{(2)} = 1) \cdot P(G_7^{(1)} = 2|X_7^{(1)} = 4) \cdot$$
$$P(X_7^{(2)} = 2|X_6^{(2)} = 5, X_7^{(2)} = 1, X_7^{(1)} = 4) \cdot P(G_7^{(2)} = 2|X_7^{(2)} = 1)$$
$$= E_L(X_7^{(1)} - S_7^{(1)}) \cdot E_G(X_7^{(1)} - G_7^{(1)}) \cdot E_N(X_7^{(2)} - S_7^{(2)}) \cdot E_G(X_7^{(2)} - G_7^{(2)})$$
$$= E_L(4-2) \cdot E_G(4-2) \cdot E_N(1-2) \cdot E_G(1-2)$$
$$= E_L(2) \cdot E_G(2) \cdot E_N(-1) \cdot E_G(-1)$$
$$= 0.05 \cdot 0.07 \cdot 0.2 \cdot 0.15 = 0.000105 = 105 \cdot 10^{-6}$$

(ii) Suppose both cars' cell phones died so you only get the observations $G_7^{(1)} = 2, G_7^{(2)} = 2$ What is the weight of each particle?

Particle	Weight
$(X_7^{(1)} = 2, X_7^{(2)} = 2)$	0.25
$(X_7^{(1)} = 4, X_7^{(2)} = 1)$	0.0105

Solution:

$$P(G_7^{(1)} = 2|X_7^{(1)} = 2) \cdot P(G_7^{(2)} = 2|X_7^{(2)} = 2)$$
$$= E_G(X_7^{(1)} - G_7^{(1)}) \cdot E_G(X_7^{(2)} - G_7^{(2)})$$
$$= E_G(2-2) \cdot E_G(2-2)$$
$$= E_G(0) \cdot E_G(0)$$
$$= 0.5 \cdot 0.5 = 0.25$$

$$P(G_7^{(1)} = 2|X_7^{(1)} = 4) \cdot P(G_7^{(2)} = 2|X_7^{(2)} = 1)$$
$$= E_G(X_7^{(1)} - G_7^{(1)}) \cdot E_G(X_7^{(2)} - G_7^{(2)})$$
$$= E_G(4-2) \cdot E_G(1-2)$$
$$= E_G(2) \cdot E_G(-1)$$
$$= 0.07 \cdot 0.15 = 0.0105$$

(c) To decouple this question, assume that you got the following weights for the two particles.

Particle	Weight
$(X_7^{(1)} = 2, X_7^{(2)}) = 2$	0.09
$(X_7^{(1)} = 4, X_7^{(2)}) = 1$	0.01

What is the belief for the location of car 1 and car 2 at $t = 7$?

Location	$P(X_7^{(1)})$	$P(X_7^{(2)})$
$X_7^{(i)} = 1$	0	$\frac{0.01}{0.09+0.01}$
$X_7^{(i)} = 2$	$\frac{0.09}{0.09+0.01}$	$\frac{0.09}{0.09+0.01}$
$X_7^{(i)} = 4$	$\frac{0.1}{0.09+0.01}$	0