1 Utilities

1. Consider a utility function of \( U(x) = 2x \). What is the utility for each of the following outcomes?
   a) 3
   b) \( L(\tfrac{2}{3}, 3; \tfrac{1}{3}, 6) \)
   c) \(-2\)
   d) \( L(0.5, 2; 0.5, L(0.5, 4; 0.5, 6)) \)

2. Consider a utility function of \( U(x) = x^2 \). What is the utility for each of the following outcomes?
   a) 3
   b) \( L(\tfrac{2}{3}, 3; \tfrac{1}{3}, 6) \)
   c) \(-2\)
   d) \( L(0.5, 2; 0.5, L(0.5, 4; 0.5, 6)) \)

3. What is the expected monetary value (EMV) of the lottery \( L(\tfrac{2}{3}, 3; \tfrac{1}{3}, 6) \)?

4. For each of the following types of utility function, state how the utility of the lottery \( U(L) \) compares to the utility of the amount of money equal to the EMV of the lottery, \( U(EMV(L)) \). Write \(<\), \(>\), \(=\), or \(\text{?}\) for can’t tell.
   a) \( U \) is an arbitrary function.
   b) \( U \) is monotonically increasing and its rate of increase is increasing (its second derivative is positive).
   c) \( U \) is monotonically increasing and linear (its second derivative is zero).
   d) \( U \) is monotonically increasing and its rate of increase is decreasing (its second derivative is negative).
2 Minimax and Expectimax

In this problem, you will investigate the relationship between expectimax trees and minimax trees for zero-sum two player games. Imagine you have a game which alternates between player 1 (max) and player 2. The game begins in state s0, with player 1 to move. Player 1 can either choose a move using minimax search, or expectimax search, where player 2’s nodes are chance rather than min nodes.

1. Draw a (small) game tree in which the root node has a larger value if expectimax search is used than if minimax is used, or argue why it is not possible.

2. Draw a (small) game tree in which the root node has a larger value if minimax search is used than if expectimax is used, or argue why it is not possible.

3. Under what assumptions about player 2 should player 1 use minimax search rather than expectimax search to select a move?

4. Under what assumptions about player 2 should player 1 use expectimax search rather than minimax search?

5. Imagine that player 1 wishes to act optimally (rationally), and player 1 knows that player 2 also intends to act optimally. However, player 1 also knows that player 2 (mistakenly) believes that player 1 is moving uniformly at random rather than optimally. Explain how player 1 should use this knowledge to select a move. Your answer should be a precise algorithm involving a game tree search, and should include a sketch of an appropriate game tree with player 1’s move at the root. Be clear what type of nodes are at each play and whose turn each play represents.

3 Lotteries in Ghost Kingdom

Diverse Utilities. Ghost-King (GK) was once great friends with Pacman (P) because he observed that Pacman and he shared the same preference order among all possible event outcomes. Ghost-King, therefore, assumed that he and Pacman shared the same utility function. However, he soon started realizing that he and Pacman had a different preference order when it came to lotteries and, alas, this was the end of their friendship.

Let Ghost-King and Pacman’s utility functions be denoted by $U_{GK}$ and $U_P$ respectively. Assume both $U_{GK}$ and $U_P$ are guaranteed to output non-negative values.

1. Which of the following relations between $U_{GK}$ and $U_P$ are consistent with Ghost King’s observation that $U_{GK}$ and $U_P$ agree, with respect to all event outcomes but not all lotteries?
   a) $U_P = aU_{GK} + b \quad (0 < a < 1, b > 0)$
   b) $U_P = aU_{GK} + b \quad (a > 1, b > 0)$
   c) $U_P = U_{GK}^2$
2. In addition to the above, Ghost-King also realized that Pacman was more risk-taking than
him. Which of the relations between $U_{GK}$ and $U_P$ are possible?

a) $U_P = aU_{GK} + b \quad (0 < a < 1, b > 0)$
b) $U_P = aU_{GK} + b \quad (a > 1, b > 0)$
c) $U_P = U_{GK}^2$
d) $U_P = \sqrt{U_{GK}}$

Guaranteed Return. Pacman often enters lotteries in the Ghost Kingdom. A particular Ghost
vendor offers a lottery (for free) with three possible outcomes that are each equally likely: winning
$1, $4, or $5.

Let $U_P(m)$ denote Pacman’s utility function for $m$. Assume that Pacman always acts rationally.

1. The vendor offers Pacman a special deal - if Pacman pays $1, the vendor will manipulate
the lottery such that Pacman **always gets the highest reward possible**. For which of these
utility functions would Pacman choose to pay the $1 to the vendor for the manipulated
lottery over the original lottery? (Note that if Pacman pays $1 and wins $m in the lottery,
his actual winnings are $m-1.)

a) $U_P(m) = m$
b) $U_P(m) = m^2$

2. Now assume that the ghost vendor can only manipulate the lottery such that Pacman
**never gets the lowest reward** and the remaining two outcomes become equally likely. For
which of these utility functions would Pacman choose to pay the $1 to the vendor for the
manipulated lottery over the original lottery?

a) $U_P(m) = m$
b) $U_P(m) = m^2$